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LEAF INVARIANTS FOR FOLIATIONS AND THE VAN EST ISOMORPHISM

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Introduction

In [5], Haefliger defined a K-fibré, G-feuilleté and gave a classifying space $B(G,K)$ for such objects. He also defined a map ϕ_H from $H^*(\underset{\sim}{g},k)$ to $H^*(B(G,K))$ which is injective for G a Lie group and K a compact subgroup. ($H^*(\underset{\sim}{g},k)$ denotes the K-basic Lie algebra cohomology of $\underset{\sim}{g}$, the Lie algebra of G.) In the special case where K is a maximal compact subgroup, $H(\underset{\sim}{g},k)$ is isomorphic to the continuous cohomology $H^*_c(G)$ of G by the Van Est Theorem [15]. In this paper we give a specific map $\Phi_G: H(\underset{\sim}{g},K) \to H^*_c(G)$ (defined in fact at the cochain level) which realizes the Van Est isomorphism, and show that $\Phi_H = \pi^* \circ r \circ \Phi_G$ where $r: H^*_c(G) \hookrightarrow H^*(G) = H^*(BG_0)$ is the inclusion, G_0 is G with the discrete topology, and $\pi: B(G,K) \to BG_0$ is the map which classifies the G_0 structure of the K-fibré, G-feuilleté.

The map Φ_H above is also shown to be related to invariants $R: H(\underset{\sim}{g},K) \to H^*(L)$ for a leaf L of a foliation, defined by Reinhart and Goldman in [11] and [4]. This is done by relating them both to the characteristic homomorphism φ_σ defined by Kamber and Tondeur in [8, p. 1409]. Specifically $R = \Phi_H \circ f$ where $f: L \to B(G,K)$ classifies the K-fibre, G-feuilleté given by the foliated normal bundle to L. As a result of this it is shown that the leaf invariants arise from the continuous cohomology of G by the inclusion of the linear holonomy into G. We also indicate briefly how to define global classes which give rise to these leaf invariants. One such class is the obstruction for a foliation to be volume-preserving. Finally, we give some examples of relations between leaf invariants and the exotic classes for foliations. In particular, this provides a way to obtain a result in [2] and [8, Vol. 279] on the nonvanishing of certain of these exotic classes.

1. Leaf invariants

We first review a construction of Kamber and Tondeur in [8, p. 1409] and [9, p. 68]. We then define Reinhart's leaf invariants as given in [11] and [4] for trivial normal bundle, and generalize the construction for arbitrary normal

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bundle. We conclude by showing that the two constructions give essentially the same invariants.

Let $G = Gl(k; R)$, and let $\underline{g}$ be its Lie algebra. Let L be a leaf of a smooth foliation $\mathscr{F}$ of codimension k, and $\pi_1(L) \to G$, the linear holonomy of L.

Let $\Gamma \subset G$ be the image of this homomorphism, and $\tilde{L}$ the covering space associated to Γ. We set

$$\tilde{L} \underset{\Gamma}{\times} G = \tilde{L} \times G/(l, g) \sim (\gamma \cdot l, \gamma \cdot g) \qquad \text{for } \gamma \in \Gamma,\ l \in \tilde{L},\ g \in G .$$

The projection π onto the first factor is the principal normal G-bundle ν of the leaf L in the foliation $\mathscr{F}$. This bundle is a discrete principal G-bundle over L. For such a bundle there is a characteristic homomorphism φ_σ defined as follows: For a compact subgroup K of G, Γ acts on G/K by left multiplication and we get a factoring of π:

$$\begin{array}{ccc} \tilde{L} \underset{\Gamma}{\times} G & \xrightarrow{\ \pi_k\ } & \tilde{L} \underset{\Gamma}{\times} G/K \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \tilde{\pi}} & \\ L & & \end{array}$$

Now assume that ν has a K reduction as a G-bundle. Then $\tilde{\pi}$ has a section $\sigma: L \to \tilde{L} \underset{\Gamma}{\times} G/K$. Let $\bigwedge^*(\underline{g}; K) = \{\omega \in A^*(G/K);\ L_g^*\omega = \omega \text{ for all } g \text{ in } G\}$ where A^* denotes differential forms, and L_g the left multiplication by g. Let $\omega \in \bigwedge^*(\underline{g}; K)$ and consider

$$\begin{array}{ccc} \tilde{L} \times G/K & \xrightarrow{\ \pi_2\ } & G/K \\ \downarrow & & \\ \tilde{L} \underset{\Gamma}{\times} G/K . & & \end{array}$$

Then $\pi_2^*\omega$ projects to a form $\tilde{\omega} \in A^*(\tilde{L} \underset{\Gamma}{\times} G/K)$ and $\sigma^*\tilde{\omega} \in A^*(L)$. The cochain map $\omega \to \sigma^*\tilde{\omega}$ induces a map $H^*(\underline{g}; K) \xrightarrow{\varphi_\sigma} H^*_{DR}(L)$, where H^*_{DR} denotes the de Rham cohomology of the manifold L, which we call the characteristic homomorphism φ_σ of L. In general φ_σ depends on σ; however if G/K is contractible then all sections are homotopic and φ_σ is independent of σ.

For $K = \{e\}$, φ_σ is the Reinhart map, as shown by the following: Since ν is a trivial G-bundle, there are global differential 1-forms $\omega_1, \cdots, \omega_k$ defined on a tubular neighborhood N of L which define $\mathscr{F}$ on N, and 1-forms η_{ij} such that

$$d\omega_{ij} = \sum_{j=1}^{k} \eta_{ij} \wedge \omega_j$$

or, in matrix notation, $d\omega = \eta \wedge \omega$. Since $\omega|_L = 0$, it follows that $d\eta|_L = \eta \wedge \eta|_L$, where the notation $|_L$ denotes the pullback to the submanifold L.

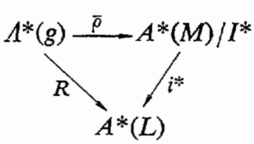
Let $\{\theta_{ij}\}$, $1 \leq i, j \leq k$, be a left invariant basis for $\bigwedge^1 (g^*)$, i.e., Maurer-Cartan forms. Then $d\theta = \theta \wedge \theta$, and the map $\theta_{ij} \to \eta_{ij}$ extends to a multiplicative cochain map $\bigwedge^* (g) \to A^*(L)$. The induced map $H^*(g) \xrightarrow{R_\omega} H^*(L)$ is the one defined by Reinhart [11].

Proposition 1.1. *If* $\omega_1, \cdots, \omega_k$ *and* σ *define the same trivializations, then* $R_\omega = \varphi_\sigma$.

Proof. It is well known [1], [5] that η is characterized by being the matrix of connection 1-forms for a Bott connection of ν with respect to the global frame $\omega_1, \cdots, \omega_k$. On the principal G-bundle associated to ν, over L, a Bott connection can be given by the connection whose horizontal subspaces are tangent to the leaves of the foliation on $\tilde{L} \underset{\Gamma}{\times} G$. Therefore, given an open covering $\{V_\alpha\}$ of L which trivializes $\tilde{L} \underset{\Gamma}{\times} G$ as a Γ-bundle, we have that the connection form on $V_\alpha \times G$ can be given by pulling back the Maurer-Cartan forms on G by the projection $V_\alpha \times G \xrightarrow{\pi_\alpha} G$. Clearly $\pi_\alpha^* \theta_{ij} = \pi_\beta^* \theta_{ij}$ because the θ_{ij}'s are left invariant, and π_α and π_β differ by an element of Γ. Let $\tilde{\theta}_{ij}$ represent the resulting global connection form on $\tilde{L} \underset{\Gamma}{\times} G$. Hence, if $\sigma : L \to \tilde{L} \underset{\Gamma}{\times} G$ represents the trivialization $\omega_1, \cdots, \omega_k$, we have that $\sigma^*(\tilde{\theta}_{ij})$ gives the matrix of connection 1-forms with respect to the global frame $\omega_1, \cdots, \omega_k$. Therefore $\eta_{ij} = \sigma^*(\tilde{\theta}_{ij})$. The result follows from this.

It is also straightforward to define R for the case of a K-reduction of the normal bundle ν, for arbitrary compact K, using differential forms [5], [4]. For this, one considers the pullback foliation on the total space of the K-bundle over a neighborhood of L, constructs the map R_ω there, for the canonical frame ω, and the K-basic forms $\bigwedge^* (g, K)$ will project to the base, giving $R : H^*(g, K) \to H^*(L)$. This map is also seen to agree with ϕ_σ.

Using the differential form construction, we are able to give a global interpretation of these classes. If the normal bundle to the foliation $\mathscr{F}$ on the manifold M is trivial, choose global ω_i's (defining the foliation) and η_{ij}'s such that $d\omega = \eta \wedge \omega$. Then we get a map $\rho : \bigwedge^* (g) \to A^*(M)$ which is not a chain map since $d\eta \neq \eta \wedge \eta$ on M. However, if we let I^* be the (differential) ideal of forms generated by the ω_i's (i.e., forms vanishing on leaves) and $A^*(M)/I^*$ the quotient, then ρ projects to a chain map $\bar{\rho}$ with commutative diagram:



Thus the leaf invariants, for any leaf, come from elements of $H^*(A^*(M)/I^*)$. The associated long exact sequence

$$\cdots \to H^{n-1}(A^*(M)/I^*) \to H^n(I^*) \to H^n_{DR}(M) \to H^n(A^*(M)/I^*) \to \cdots$$

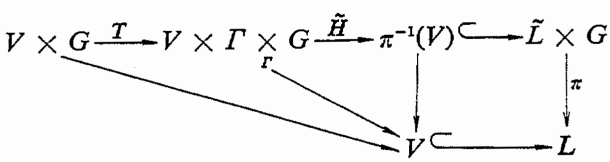
is discussed in Reinhart [10]. From this, for example, we can define $\operatorname{tr}(\eta) \in H^1(A^*(M)/I^*)$ which depends only on the foliation, and is the zero class if and only if the foliation globally preserves a volume. In contrast, $i^*(\operatorname{tr}\eta) \in H^1_{DR}(L)$ is zero if and only if the linearized holonomy is volume-preserving; see [13].

2. Haefliger's characteristic homomorphism

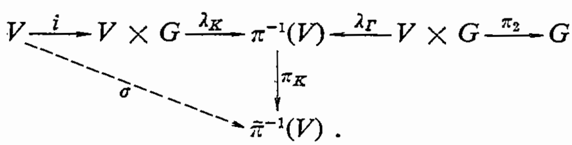
In [5], Haefliger defined the notion of a K-fibré, G-feuilleté on a manifold L, for general G, and a characteristic homomorphism $\phi_H : H^*(\underline{g}; K) \to H^*_{DR}(L)$. A discrete G-bundle with a given reduction to a K-bundle is an example of a K-fibré G-feuilleté.

Proposition 2.1. *Given $G = Gl(k; R)$, K a compact subgroup, and a K-fibré G-feuilleté on L with a K-reduction defined by a section σ of $\tilde{\pi}$, (of § 1), then $\phi_H = \phi_\sigma$.*

Proof. The bundle $\tilde{L} \times_\Gamma G \xrightarrow{\pi} L$ (i.e., ν) has a natural Γ reduction defined as follows. Let $\tilde{L} \xrightarrow{P} L$ be the covering space associated to Γ and $V \subset L$ be such that $V \times \Gamma \xrightarrow[H]{\approx} p^{-1}(V)$ is an isomorphism. Then we have



where $T(v, g) = (v, [e, g])$, $T^{-1}(v, [\gamma, g]) = (v, \gamma^{-1}g)$, and $\tilde{H}(v, [\gamma, g]) = [H(v, \gamma)^{-1}, g]$. Then $\lambda_\Gamma = \tilde{H} \circ T$ is the required trivialization over V. Now let $\lambda_K : V \times G \to \pi^{-1}(V)$ be a K-trivialization over V. Thus the λ_Γ's, for various V, differ by elements of Γ and the λ_K's differ by elements of K. Now consider

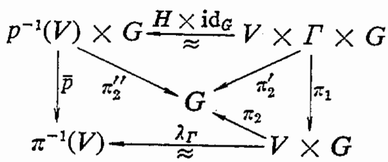


Let the composite of the top row be h. Note that h^* is Haefliger's map ϕ_H on V; see [5]. The maps $\pi_K \circ \lambda_K \circ i$ agree on overlaps of open sets V (since the

λ_K's differ by elements of K) and hence fit together to define a global section σ of $\tilde{\pi}$. Let $\pi_{K*}: A^*(\tilde{L} \times_\Gamma G)_{K\text{-basic}} \to A^*(\tilde{L} \times_\Gamma G/K)$ denote projection of K-basic forms; then

$$h^* = i^* \circ \lambda_K^* \circ \lambda_\Gamma^{-1*} \circ \pi_2^* = \sigma^* \circ \pi_{K*} \circ \lambda_\Gamma^{-1*} \circ \pi_2^* , \tag{2.1}$$

since $\sigma^* \circ \pi_{K*} = i^* \circ \lambda_K^*$ on K-basic forms. Then by the commutative diagram



we get $\bar{p}^* \circ \lambda_\Gamma^{-1*} \circ \pi_2^* = \pi_2''^*$, and by tracing through the definition of φ_σ we find that the expression in (2.1) is φ_σ. Thus Proposition 2.1 is proved.

3. The cochain map inducing the Van Est isomorphism

In this section, G denotes a connected semi-simple Lie group, and K a maximal compact subgroup.

Let $[g] = (g_0, \cdots, g_n)$ be an element of $G^{n+1} = G \times \cdots \times G$, $(n + 1)$ times. $L_g[g]$ will denote the $(n + 1)$-tuple $(gg_0, \cdots, gg_n)$, and $[g]_i$ the n-tuple $(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_n)$. The coset of g in G/K will be denoted $\bar{g}$, and $[\bar{g}]$ will denote the image of $[g]$ in $(G/K)^{n+1}$. Let $[t] = (t_1, \cdots, t_n)$ be an element of R^n, and let Δ^n denote the n-simplex given by

$$\Delta^n = \left\{[t] \in R^n \,|\, 0 \le t_i \le 1 , \sum_{i=1}^{n} t_i \le 1\right\} .$$

For $i \neq 0$, the ith vertex is $(0, \cdots, 1, 0, \cdots, 0)$ with 1 in the ith position, and for $i = 0$ it is $(0, \cdots, 0)$. Let $F_i: \Delta^{n-1} \to \Delta^n$ be the inclusion of Δ^{n-1} as the ith face of Δ^n, that is, $F_i(t_1, \cdots, t_{n-1}) = (t_1, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1})$.

Proposition 3.1. *For each $n \ge 0$, there is a map $\sigma^n: \Delta^n \times G^{n+1} \to G/K$ with the following properties*:

(1) *σ^n is differentiable.*

(2) *$\sigma^n([t], L_g \cdot [g]) = L_g \cdot \sigma([t], [g])$, where $L_g \cdot \sigma([t], [g])$ denotes the action of G on G/K by the left multiplication.*

(3) *$\sigma^n(F_i([t], [g])) = \sigma^{n-1}([t], [g]_i)$, for $[t] \in \Delta^{n-1}$ and $[g] \in G^{n+1}$.*

(4) *By fixing $[g] \in G^{n+1}$ we get a map which we will denote by $\sigma^n_{[g]}: \Delta^n \to G/K$. The map $\sigma^n_{[g]}$ is a diffeomorphism onto its image and sends the ith vertex of Δ^n to $\bar{g}_i$.*

Proof. Let $\underline{k} \oplus \underline{p}$ denote the Cartan decomposition of $\underline{g}$, corresponding to the polar decomposition $G = K \times P$. Then G/K can be identified with P, and the tangent space $T_{\bar{e}}(G/K)$ with $\underline{p}$. Since exp: $\underline{p} \to P$ is a diffeomorphism, we can consider the maps exp and log as diffeomorphisms between $T_{\bar{e}}(G/K)$ and G/K. The diffeomorphism exp determines a unique path joining $\bar{e}$ to any other given point of G/K. We can left translate these paths in order to define paths joining any two given points of G/K; these paths on G/K are well defined and unique because $k(\exp x)k^{-1} = \exp(\mathrm{Ad}(k)x)$, for all k in K and x in $\underline{p}$. These paths give rise to a join operation on G/K. For a fixed $[g]$ in G^{n+1} we use this join operation to define simplices inductively on G/K. For vertices $(\bar{g}_0, \cdots, \bar{g}_n)$ we "fill-in" the simplex by connecting $\bar{g}_n$ to each point in the simplex with vertices $(\bar{g}_0, \cdots \bar{g}_{n-1})$ using the above paths.

Precisely, maps $\sigma^n_{[g]}: \Delta^n \to G/K$ are defined as follows:

For $n = 0$, $\sigma^0_{(g_0)}(0) = \bar{g}_0$, and for $n = 1$, $\sigma^1_{(g_0, g_1)}(t_1) = L_{g_0} \cdot \exp((1 - t_1) \log \overline{g_0^{-1} g_1})$, In general we define inductively,

$$\sigma^n_{[g]}(t_1, \cdots, t_n) = L_{g_0} \cdot \exp((1 - t_1) \log \sigma^{n-1}_{L_{g_0^{-1}}[g]_0}(t_2, \cdots, t_n)) . \tag{3.1}$$

It is clear that σ^n is differentiable. The properties (2), (3) and (4) of σ^n can all be verified inductively by straightforward computations using (3.1).

Let Γ be a group with the discrete topology. We recall the simplicial construction of the space $B\Gamma$ which classifies principal Γ-bundles. For each $n \geq 0$, take a disjoint union of n-simplices indexed by the elements of Γ^{n+1}, and identify $([t], [\gamma]_i) \in \Delta^{n-1} \times \Gamma^n$ with $(F_i[t], [\gamma]) \in \Delta^n \times \Gamma^{n+1}$, for $[t] \in \Delta^{n-1}$ and $[\gamma] \in \Gamma^{n+1}$. The resulting acyclic simplicial complex is denoted $E\Gamma$. For $\gamma \in \Gamma$ we have the left action on Γ^{n+1} given by $L_\gamma(\gamma_0, \cdots, \gamma_n) = (\gamma\gamma_0, \cdots, \gamma\gamma_n)$, which induces a free discontinuous action of Γ on $E\Gamma$ by permuting the simplices. The quotient space of this Γ action is $B\Gamma$ and it has a simplicial structure with ordered simplices induced from $E\Gamma$. The real simplicial n-cochains $C^n(\Gamma)$ on $B\Gamma$ consist of the set of all functions from Γ^{n+1} to the reals with the property that if $f \in C^n(\Gamma)$, then $f(\gamma\gamma_0, \cdots, \gamma\gamma_n) = f(\gamma_0, \cdots, \gamma_n)$. The coboundary $\delta^n: C^n(\Gamma) \to C^{n+1}(\Gamma)$ is given by $\delta^n f(\gamma_0, \cdots, \gamma_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(\gamma_0, \cdots, \hat{\gamma}_i, \cdots, \gamma_{n+1})$. The cohomology of this cochain complex is called the cohomology of the group Γ with real coefficients. It will be denoted $H^*(\Gamma)$. This construction was described in [3].

Suppose that Γ is a subgroup of G. Then we can restrict $\sigma^n: \Delta^n \times G^{n+1} \to G/K$ to obtain $\sigma^n: \Delta^n \times \Gamma^{n+1} \to G/K$.

Proposition 3.2. *The maps $\sigma^n: \Delta^n \times \Gamma^{n+1} \to G/K$ for $n \geq 0$ define a continuous map $\sigma: E\Gamma \to G/K$ satisfying*

(1) *σ is differentiable when restricted to any simplex of $E\Gamma$.*

(2) *σ is equivariant with respect to the left actions of Γ on $E\Gamma$ and G/K respectively.*

Proof. Given an n-simplex of $E\Gamma$ corresponding to $[\gamma]$ we map it into G/K by $\sigma^n_{[\gamma]}$. This map is differentiable by Proposition 3.1. These maps agree with the identifications of these simplices in $E\Gamma$ because of Proposition 3.1 (3), and hence yield a map $\sigma: E\Gamma \to G/K$. Since $E\Gamma$ has the weak topology as a simplicial complex, it follows that σ is continuous. The map σ is equivariant because of Proposition 3.1 (2).

We will make use of the de Rham theory for simplicial complexes as developed by Sullivan [14]. Let $|X|$ denote a simplicial complex. A simplicial differential form on $|X|$ is a choice of an ordinary smooth differential form on each closed simplex which satisfies the following compatibility condition. If Δ is the intersection of two simplices, then the form pulled back to Δ from one of the simplices should equal the form pulled back to Δ from the other. The ordinary exterior derivative on each simplex induces a differential on simplicial differential forms. The complex of these real valued simplicial differential forms with exterior derivative will be denoted by $\tilde{A}^*(|X|)$, and the resulting cohomology by $\tilde{H}^*_{D.R.}(|X|)$. There is a map $\rho: \tilde{A}^*(|X|) \to C^*(|X|, R)$ given by $\rho(\varphi)(\Delta^n) = \int_{\Delta^n} \varphi$, where $C^*(|X|, R)$ are the real cochains on $|X|$. The map ρ commutes with differentials by Stokes' theorem and induces

$$\rho: \tilde{H}^*_{D.R.}(|X|) \to H^*(|X|, R) .$$

In particular, for $|B\Gamma|$, we get

$$\rho: \tilde{H}^*_{D.R.}(|B\Gamma|) \to H^*(B\Gamma|, R) \approx H^*(\Gamma) .$$

Proposition 3.3. *The function* $\phi_\Gamma: \bigwedge^* (\underline{g}, K) \to \tilde{A}^*(|B\Gamma|)$ *defined by* $\phi_\Gamma(\omega)(\gamma_0, \cdots, \gamma_n) = \sigma^*_{[\gamma]}\omega$ *yields a map of complexes.*

Proof. $\phi(\omega)(\gamma\gamma_0, \cdots, \gamma\gamma_n) = \phi(\omega)(\gamma, \cdots, \gamma_n)$ since

$$\sigma^*_{(\gamma\gamma_0,\cdots,\gamma\gamma_n)}\omega = \sigma^*_{(\gamma_0,\cdots,\gamma_n)} \cdot L^*\omega = \sigma^*_{(\gamma_0,\cdots,\gamma_n)}\omega$$

by Proposition 3.1 (2), and since ω is left invariant. $\phi(\omega)$ is a simplicial differential form because $F^*_i\sigma^*_{[g]}\omega = \sigma^{n-1*}_{[g]_i}\omega$ by Proposition 3.1 (3). Therefore we get a map $\Phi_\Gamma: H(\underline{g}, K) \to H^*(\Gamma)$, where $\Phi_\Gamma = \rho \circ \phi_\Gamma$ for Γ a subgroup of G. Let G_0 denote G with the discrete topology. The subcomplex $C^n_c(|BG_0|, R)$ of $C^n(|BG_0|, R)$ consisting of those cochains $f: G_0^{n+1} \to R$ which are continuous with respect to the Lie group topology on G are called the continuous cochains, the cohomology of which is denoted $H^*_c(G)$.

Proposition 3.4. *The image of* $\Phi_{G_0}: H^*(\underline{g}, K) \to H^*(G_0)$ *is contained in* $H^*_c(G)$.

Proof. This follows from the differentiability of σ^n and the fact that ϕ_{G_0} is defined in terms of σ.

Let us denote by Φ_G the map from $H^*(\underline{g}, K)$ to $H^*_c(G)$ which is induced by Φ_{G_0}. As a corollary to Proposition 3.4, we have

Corollary 3.1. *Let i denote the inclusion of Γ in G_0. Then* $\phi_\Gamma: \bigwedge^k (\underline{g}, K) \to \tilde{A}^*(|B\Gamma|)$ *factors as* $i^* \circ \phi_{G_0}$ *and consequently* $\Phi_\Gamma: H^*(\underline{g}, K) \to H^*(\Gamma)$ *factors* *as* $i^* \circ \Phi_G$ *where* $i^* : H^n_c(G) \to H^n(\Gamma)$ *is given by restricting to* Γ^{n+1} *the continuous n-cochains on* G_0.

It was shown by Van Est [15] that $H^*(\underline{g}, K)$ and $H^*_c(G)$ are isomorphic; however an explicit isomorphism was not given.

Proposition 3.5. $\Phi_G : H^*(\underline{g}, K) \to H^*_c(G)$ *is an algebra isomorphism.*

Proof. One way to see this is to note that Φ_c is induced by a mapping of continuously injective resolutions of the reals in the sense of Hochschild and Mostow ([6], see the proof of Theorem 6.1). However, we will show directly that Φ_c is injective, and then it will follow that Φ_c is onto from the fact that they are isomorphic and the finite dimensionality of $H^*(\underline{g}, K)$. Let Γ be a discrete subgroup of G such that $\Gamma \backslash G/K$ is a compact orientable manifold. The mapping $\sigma : E\Gamma \to G/K$ is Γ equivariant and hence induces a mapping $\sigma : B\Gamma \to \Gamma \backslash G/K$. Since both $E\Gamma$ and G/K are contractible, we conclude that σ is a homotopy equivalence. Consider the following diagram which is easily seen to commute:

$$\begin{array}{ccccc} H^*(\Gamma) & \xleftarrow{i^*} & H^*_c(G) & \xleftarrow{\Phi_G} & H(\underline{g}, K) \\ \rho \uparrow & & & & \downarrow j \\ H^*_{DR}(|B\Gamma|) & & \xleftarrow{\quad \sigma^* \quad} & & H^*_{DR}(\Gamma \backslash G/K) \end{array} ,$$

where j is the projection of the left invariant forms on G/K to $\Gamma \backslash G/K$. Since σ is a homotopy equivalence, $\rho \circ \sigma^*$ is an isomorphism. The mapping j is injective (see [7, Lemma 4.21, p. 22]). Hence Φ_G is injective and hence an isomorphism. Φ_G is an isomorphism of real algebras because all the other maps in the diagram are mappings of real algebras.

4. The simplicial Van Est map, leaf invariants, and Φ_G

The construction in § 1 of φ_σ, which gives a characteristic homomorphism for a flat Γ-bundle over L, can be generalized to the case where L is any simplicial complex.

We will outline this construction first for the case of the universal Γ-bundle over the simplicial complex $|B\Gamma|$. The G/K bundle associated to the universal Γ-bundle is $E\Gamma \times_\Gamma G/K \xrightarrow{\pi} B\Gamma$. This bundle restricted to a closed simplex Δ in $|B\Gamma|$ is diffeomorphic to $\Delta \times G/K$. This trivialization of $E\Gamma \times_\Gamma G/K$ can be chosen to be a Γ-trivialization. There is a map from $A^*(G/K)$ to $A^*(\Delta \times G/K)$ given by projection of $\Delta \times G/K$ to G/K. These maps are compatible in the sense that if Δ' is contained in Δ, then the map to $A^*(\Delta' \times G/K)$ is the same as the map to $A^*(\Delta \times G/K)$ followed by restriction to $A^*(\Delta' \times G/K)$.

There is a section $|\sigma| : B\Gamma \to E\Gamma \times_\Gamma GK$ given by $|\sigma|(x) = (\tilde{x}, \sigma(\tilde{x}))$, where

$\tilde{x} \in E\Gamma$ projects to x and $\sigma\colon E\Gamma \to G/K$ is the map defined in Proposition 3.2. $|\sigma|$ is well defined because of Proposition 3.2 (2), and the restriction of $|\sigma|\colon \Delta \to \Delta \times G/K$ for Δ in $|B\Gamma|$ is differentiable by Proposition 3.2 (1). The composite of $|\sigma|\colon \Delta \to \Delta \times G/K$ followed by projection to G/K induces a map $A^*(G/K) \to A^*(\Delta)$. Because of the compatibility of the maps $A^*(G/K) \to A^*(\Delta \times G/K)$ we have the following proposition.

Proposition 4.1. *The section* $|\sigma|$ *induces* $\phi_{|\sigma|}\colon \bigwedge^*(\underline{g}, K) \to \tilde{A}^*(|B\Gamma|)$ *which in turn induces* $\phi_{|\sigma|}\colon H(\underline{g}, K) \to \tilde{H}_{DR}(|B\Gamma|)$.

We set $\Phi_{|\sigma|} = \rho \circ \phi_{|\sigma|}\colon H(\underline{g}, K) \to H^*(|B\Gamma|, R) \approx H^*(\Gamma)$.

Proposition 4.2. $\phi_{|\sigma|}$ *is the same as the map* ϕ_Γ *given by Proposition* 3.3, *and hence* $\Phi_{|\sigma|} = \Phi_\Gamma$.

Proof. This follows simplex by simplex from the definitions.

For a Γ-bundle over a simplicial complex $|L|$, there is a simplicial map $j\colon |L| \to |B\Gamma|$ which fits into a commutative diagram of Γ-bundles:

$$\begin{array}{ccc} \tilde{L} \underset{\Gamma}{\times} G/K & \xrightarrow{\ \tilde{j}\ } & E\Gamma \underset{\Gamma}{\times} G/K \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow \\ L & \xrightarrow{\ \ j\ \ } & B\Gamma \end{array}$$

Using $\tilde{j}$ we can map $A^*(G/K)$ into $\pi^{-1}(\Delta)$, for Δ a simplex in $|L|$, and analogously with the construction of $\phi_{|\sigma|}$, we can define $\phi_{|s|}\colon H^*(\underline{g}, K) \to \tilde{H}^*_{DR}(|L|)$ where $|s|$ is any smooth simplicial section of $\tilde{\pi}$, (that is, one which is differentiable when restricted to each simplex of $|L|$). Such a section is given by $|s| = \tilde{j}^{-1} \circ |\sigma| \circ j$. Any two such sections are homotopic since G/K is contractible, and the homotopy can be taken to be differentiable when restricted to any simplex in $|L|$. Therefore $\phi_{|s|} = j^* \circ \phi_{|\sigma|}$ for any such $|s|$. Furthermore $j^*\colon \tilde{H}^*_{DR}(|B\Gamma|) \to \tilde{H}^*_{DR}(|L|)$ is independent of the choice of j since all such choices are simplicially homotopic.

From the above and Corollary 3.1, we have

Proposition 4.3. $\phi_{|s|}\colon H^*(\underline{g}, K) \to \tilde{H}^*_{DR}(|L|)$ *factors as* $\phi_{|s|} = j^* \circ i^* \circ \phi_{G_0}$ (*where* i^* *is induced by the inclusion of* Γ *in* G) *and hence* $\Phi_{|s|} = j^* \circ i^* \circ \Phi_G$.

The above can be summarized in the following commutative diagram:

$$\begin{array}{ccccccc} H^*(|L|; R) & \xleftarrow{\ j^*\ } & H^*(\Gamma) & \xleftarrow{\ \ i^*\ \ } & H^*_c(G) & \nwarrow^{\Phi_G}_{\approx} & \\ \uparrow{\scriptstyle\rho} & & \uparrow{\scriptstyle\rho} & & \uparrow{\scriptstyle\rho} & & \\ \tilde{H}^*_{DR}(|L|) & \xleftarrow{\ j^*\ } & \tilde{H}^*_{DR}(|B\Gamma|) & \xleftarrow{\ i^*\ } & \tilde{H}^*_{DR}(|BG_0|) & \xleftarrow{\ \phi_{G_0}\ } & H^*(\underline{g}, K) \end{array}$$

Corollary 4.1. *If* Γ *is finite or is contained in a compact connected Lie subgroup of* G *then* $i^* = 0$, *and hence* $\Phi_{|s|}$ *is zero.*

Proof. The real continuous cohomology of a finite group or of a compact connected Lie group is zero [15].

Suppose now that L is a smooth manifold with a smooth triangulation $|L|$. If s is a smooth section $s: L \to \tilde{L} \times_{\Gamma} G/K$ it induces a map $\phi_s: H(\underline{g}, K) \to H_{DR}(L)$, and when we consider s as a smooth simplicial section $|s|$ we get $\phi_{|s|}: H(\underline{g}, K) \to \tilde{H}_{DR}(|L|)$. It is easy to see that ϕ_s followed by the natural map of $H_{DR}(L)$ into $\tilde{H}_{DR}(|L|)$ is the same as $\phi_{|s|}$. Furthemore, by [14] the composite of the map of $H_{DR}(L)$ into $H_{DR}(|L|)$ followed by ρ yields the usual de Rham isomorphism. Thus we have

Theorem 4.1. *For L a manifold the map $\Phi_s: H^*(\underline{g}, K) \to H^*(L; R)$ is the same as $j^* \circ i^* \circ \Phi_G$, where $j: L \to B\Gamma$ classifies the Γ-bundle over L, and i is the inclusion of Γ in G.*

In [5] Haefliger gave a classifying space $B(G, K)$ for a K-fibré, G-feuilleté. He also defined a map $\phi_H: H^*(\underline{g}, K) \to H^*(B(G, K); R)$, corresponding to ϕ_H in § 2. $B(G, K)$ can be taken to be $E(G_0) \times_{G_0} G/K$. Let $\pi: B(G, K) \to BG_0$ be the natural projection; it classifies the G_0 structure of the K-fibré, G-feuilleté. Let us take $G = Gl(n; R)$, and K a maximal compact subgroup of G. Then we have

Corollary 4.2. $\phi_H = \pi^* \circ \Phi_G$.

Proof. This follows from Proposition 2.1 and the fact that we can use Theorem 4.1 with $j = \pi$ and $i =$ identity.

We can apply Theorem 4.1 to the leaf invariants of a smooth foliation. Let Γ be the linear holonomy, $j: L \to B\Gamma$ the map which classifies the normal bundle to L as a discrete Γ-bundle, and $i: \Gamma \to G$ the inclusion. We get

Corollary 4.3. *The following diagram commutes:*

$$\begin{array}{ccc} H^*(L) & \xleftarrow{j^*} & H^*(B\Gamma) \\ \uparrow R & & \uparrow i^* \\ H^*(\underline{g}, K) & \xrightarrow{\Phi_G} & H^*_c(G) \end{array}$$

where R is the Reinhart map discussed in § 1.

Now, for example, Corollary 4.1 gives information about the map R.

5. The exotic classes

It is known that several exotic classes of foliations are nonvanishing (in $B\Gamma$). References for these are [2] and [8]. In this section we show how certain of these relate to the leaf invariants.

Let G be a semi-simple connected Lie group, H a connected subgroup of G such that G/H is compact orientable, K a maximal compact subgroup of G, $K' \subset K$ a maximal compact subgroup of H, and Γ a discrete subgroup of G

such that the spaces $\Gamma\backslash G$, $\Gamma\backslash G/K = L$, $\Gamma\backslash G/K'$ are compact orientable manifolds.

The projection $G/K \times G/H \to G/H$ defines a foliation which projects to one on $E = G/K \underset{\Gamma}{\times} G/H$, where Γ acts on the left of both factors. The exotic characteristic classes of this foliation are elements of $H^*_{DR}(E)$. We can integrate them over the fibre G/H of $E \to L$ to obtain elements of $H^*_{DR}(L)$, These elements are in the image of $\phi_\sigma : H^*(g, K) \to H^*(L)$, where ϕ_σ is the characteristic homomorphism of the discrete G-bundle $\tilde{L} \underset{\Gamma}{\times} G \to L$ of § 1. This is seen by the following commutative diagram:

$$
(5.1)\qquad
\begin{array}{ccc}
 & H^*(\mathfrak{A}_n, SO_n) & \\
\swarrow & & \searrow \\
H^*(\underset{=}{g}, K') & \longrightarrow & H^*(E) \\
\downarrow & & \downarrow{\scriptstyle I_{G/H}} \\
H^*(\underset{=}{g}, K) & \xrightarrow[\phi_\sigma]{} & H^*(\Gamma\backslash G/K)
\end{array}
$$

where the upper triangle gives the exotic classes of the foliation on E. See [2] for notation and details of this. $I_{G/H}$ denotes integration over the fibre G/H, and the left hand vertical map corresponds to integration over the fiber K/K'. As noted above, ϕ_σ is injective.

Kamber-Tondeur have computed the maps in the upper triangle for a large class of groups. See [8, Vol. 279]. For $G = Sl(n; \boldsymbol{R})$, n even, and H the subgroup fixing a ray in R^n, they obtained:

The exotic classes are of the form $h_I c_J$ where the multi-indices $J \subset \{1, 2, \cdots, n-1\}$ and $I \subset \{1, 3, \cdots, n-1\}$. Now $K = SO_n$, $K' = SO_{n-1}$ and $H^*(g, K) = E(v_3, v_5, \cdots, v_{n-1}, \chi)$ an exterior algebra on generators v_i of dimension $2i - 1$, and χ of dimension n. One then finds, by direct computation,

Proposition 5.1. *If* $\dim(c_J) = 2(n-1)$ *and* $1 \in I$, *then (up to real multiple)* $I_{G/H}(h_I c_J) = \phi_\sigma(v_{I'} \cdot \chi)$ *where* $I' = I - \{1\}$. *Thus these* $h_I c_J$ *are nonzero in* $H^*(E)$ *and hence in* $H^*(B\Gamma_k)$.

This generalizes the case for $n = 2$ in [12]. One hopes that for other groups G there will be further relationships between exotic classes and leaf invariants.

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